Iconic and Symbolic Modes of Representation in Descartes' *Geometry*

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1. Diagrams as Icons and Symbols

Geometry, Euclid tells us, is concerned first of all with certain distinguished objects, like the straight line, circle, and triangle on the plane; and with certain distinguished procedures, the use of ruler and compass to construct points and figures. To represent his investigation of the objects by means of the procedures, he uses diagrams where wholes and parts are indicated by letters; ratios and proportions, involving wholes and parts thus indicated; and natural language that expresses the inferential progress of the argument as well as the relations among the different kinds of representations. In the work of Descartes, some geometrical things are represented by algebraic equations, whose constants and variables are used to indicate parts and wholes of objects represented by diagrams, and ratios and proportions, where parts and wholes are also still indicated by letters.

Many twentieth century philosophers of mathematics argued that the representation of geometrical things by diagrams was defective or misleading. Some of them asserted that the proper way to represent geometry is through a set of axioms; then any model of an improved axiomatization of Euclidean geometry is what we mean by Euclidean geometry. However, no axiomatization has succeeded in capturing all the important features of Euclidean geometry, and no axiomatization is categorical. In

Euclid’s *Elements*, the canonicity of the straight line—and of the triangle as the simplest figure constructible by straight lines on the plane—and the circle is clear. When Hilbert, in his role as formalist, claims that all relevant geometrical information is embedded in sets of axioms, so that geometry is only what is common to all interpretations of a theory, up to isomorphism, he cannot account for the canonicity of certain objects. If “point,” “line,” and “plane” can be given alternative interpretations that yet produce a model isomorphic to Euclid's, then for Hilbert the formalist there is no reason to demur. And by the same token there is no reason to prefer a Euclidean to a non-Euclidean theory for geometry. Geometry then becomes a kind of smorgasbord of models; philosophers of the late nineteenth century were dismayed because they felt they had lost all grounds for choosing which geometry was “true,” since the appeal to (usually Kantian) intuition had been discredited as subjective, in either a psychologistic or transcendental sense.[1]

Yet Hilbert, writing in his role as geometer, acknowledged the canonicity of certain objects as a matter of course, since without appeal to that canonicity certain domains like differential geometry could not even be broached. Introducing the chapter on differential geometry in *Geometry and the Imagination*, he writes: “we will, to start with, investigate curves and surfaces only in the immediate vicinity of any one of their points. For that purpose, we compare the vicinity, or ‘neighborhood,’ of such a point with a figure which is as simple as possible, such as a straight line, a plane, a circle, or a sphere, and which approximates the curve as closely as possible in the neighborhood under consideration.” [2] For this sentence to be meaningful and useful to the mathematicians reading his book, Hilbert must assume the ability to refer to these determinate items.

Some philosophers, for example certain exponents of the Bourbaki school, argued that the proper way to represent geometry in textbooks is to identify $\mathbf{E^n}$ with $\mathbf{R^n}$ via a coordinate system. However, this identification is problematic. First, $\mathbf{R^n}$ is ambiguous: it may represent the set $\mathbf{R^n}$ (the set of all n-tuples of reals), as well as a variety of topological spaces, vector spaces, or metric spaces. If it is considered a vector space, one must choose a basis for the space; however, there is no canonical basis unless we select the vector space of n-tuples of real numbers by component-wise addition and scalar multiplication, in which case there is a canonical basis: (1, 0, 0, … 0), (0, 1, 0… 0), … (0, 0, … 1). We may then also choose the canonical inner product $(x, y) = \Sigma_{i=1}^{n} x_i y_i$, a metric that uses the norm of the vector $\mathbf{x}$, $||\mathbf{x}|| = ((\mathbf{x},\mathbf{x}))^{1/2}$: $d(x, y) = ||\mathbf{x}-\mathbf{y}||$, and a coordinate system. But there is no natural, geometrically determined choice of coordinate system, and indeed the superposition of the coordinate system can obscure the difference between underlying geometrical properties of the space and artefactual properties that stem from the coordinate system.[3]

One way to correct partially for the inadequacies of a set of axioms, or of $\mathbf{R^n}$ with a canonical inner product, norm, and metric, and an arbitrary coordinate system, as representations of n-dimensional Euclidean space and its objects and procedures, is to use diagrams in tandem with the other modes of representation. Philosophers have often objected to diagrams on the grounds that the knowledge furnished by diagrams is allegedly intuitive, but that "intuition" is a metaphysical notion (as used, for example, by Descartes or Kant) that has been discredited, since no knowledge is immediate or self-evident. If, however, we note that historically, in the work of Euclid and Descartes for example, diagrams are not used alone but in tandem with other modes of representation, then this objection loses its strength. We can admit that knowledge claims in mathematics sometimes require diagrams, without supposing that the knowledge is conferred by the diagram alone. The use of diagrams is part of mathematical experience, but it need not involve the discredited connotations of intuition.

When we look at the use of diagrams in tandem with other mode of representation, I argue that philosophically interesting features of their use appear. First, clearly, diagrams are icons. Some objects of mathematics are shapes, and many shapes cannot be adequately represented and studied without icons of them; and even shapes that elude iconic representation, like infinite-sided polygons or "monster" functions, may be indicated indirectly by analogy, often by icons or symbols with strongly iconic aspects. Second, diagrams may represent symbolically, depending on their discursive context: that is, they may represent things they do not resemble. Third, some diagrams may represent both iconically and symbolically within one and the same argument, where a determinate and contained ambiguity is exploited by the mathematician to further the argument. Finally, the interaction of iconic and symbolic modes of representation within geometrical arguments repays philosophical study. It often exhibits the confluence of different domains, each with its own traditions of representation, as well as the development of new means of expression, novelties that affect how mathematical knowledge is both justified and discovered.[4] This kind of interaction has not, for obvious reasons, attracted the attention of philosophers of mathematics whose concern is to purify mathematics and re-write it in a single idiom. In this essay, I will illustrate all of these points by looking at Descartes' *Geometry*.

2. Descartes' *Geometry* as the Exemplar of Cartesian Method

The first and second Books of Descartes' *Geometry* include diagrams in which circles are depicted, but there is something noteworthy in their depiction. For the most part they are represented not by continuous lines, but by dotted lines, as if they were ghosts of, or references to, circles instead of the circles themselves. And this is in a sense the case, for Descartes intends these circles to function either as constructing curves or as loci. As a constructing curve, a circle is something of a revenant in the house of Descartes' method and can enter only through the back door or windows; and as a constructed locus it is not all there. Although the *Geometry* inaugurates the study of higher algebraic curves, Descartes' reductionism prevents curves from coming into focus as objects of study and curtails his own investigation of them. This makes his depiction of them as symbolic as it is iconic: they are the instrument of construction (and as such their unity is not examined) or the product of construction (and as such their unity is problematic). Mathematical things become objects of knowledge for Descartes when they are constructed according to the "order of reasons," but in the case of the circle such construction presupposes the prior availability of the circle; it also involves the disparity between rational and real numbers, which Descartes however does not want to take up. [5]

In the late Middle Ages, the progression of algebra from a problem-oriented treatment to a more purely symbolic, abstract treatment carried with it a change in the conception of both its uses and the objects to which it was applied. Though originally designed for problems involving numbers, algebra also came to be used in the treatment of geometrical problems, as well as in various practical applications. As abstract notation for constants and variables was developed, algebra increasingly came to be thought of as a science of abstract magnitude in general, *mathesis universalis*. The increasing polysemy of algebra led to a change both in the conception of geometrical figure and in the conception of number, which ultimately resulted in the nineteenth century hybrid, the real number line.

Thus geometry around 1600 was undergoing a transformation. On the one hand, fundamental differences existed between arithmetical and geometrical methods and operations: in particular, multiplication and division were linked by problematic, limited, and unpromising analogies. The product of two line segments was interpreted as an area, and that of three line segments as a volume; thus multiplication was seen to involve a change of dimension and moreover could not be represented or interpreted intuitively for cases of more than three dimensions. Even Descartes treated multiplication this way in the *Regulae* (composed around 1628). Division was often interpreted as resulting in a ratio, that is, a relation rather than another magnitude. And yet, on the other hand, mathematicians like Simon Stevin and François Viète were increasingly interested in the application of algebra, as well as the use of numbers, in problems of geometrical analysis.

Descartes' *Discourse on Method* was first published in 1637 as an introduction to three mathematical and scientific treatises ("qui sont des essais de cette méthode," as the title states), and one of them was the *Geometry*. The *Discourse* includes a précis of the argument of the *Meditations on First Philosophy*; and that argument is in turn recapitulated at the beginning of the *Principles of Philosophy*. Thus Descartes carefully locates his mathematics on the one hand and his physics on the other with respect to the *Meditations*, which is prior in the methodological "order of reasons" and provides their metaphysical justification and legitimation. [6] His conception of method is intuitionist and reductionist in the following sense, as I argued in my book *Cartesian Method and the Problem of Reduction*. Cartesian method organizes items of knowledge within a domain, and domains within the sphere of human knowledge as a whole, according to an "order of reasons" which begins with self-evidently indubitable, clear and distinct ideas, and proceeds by a chain of reasoning intended to be both truth-preserving and ampliative. Another formulation of the "order of reasons" is that the unfolding of knowledge must begin with simple things, which are in themselves transparent to reason, and move on by means of a constructive procedure that brings the simple things into relation to constitute complex things. [7] Thus, it is hoped, any more mediately known, complex item can always be led back to, and indeed recovered from, the simples from which it arose. Descartes presents both the simples, and his relational means of concatenating them, as if they were obvious, unique, and transparent—not requiring further explanation; and he moreover assumes that the simples along with the complex things are homogeneous.

In one sense, these reductionist assumptions are the key to Descartes' mathematical success. According to Descartes, the simple starting points of geometry are straight line segments, as he announces at the beginning of the *Geometry*: "All the problems of Geometry can easily be reduced to such terms that knowledge of the lengths of certain straight lines is sufficient for its construction." [8] The reductionist thrust of his method makes it an admirable problem-solving device, for his next revelation is that both the product and the quotient of straight line segments ought to be interpreted as further straight line segments. His algebra of geometrical magnitudes is closed; in one brilliant insight, he has freed the algebra of magnitude from the constraints and complications that hampered Viète. His construction procedure (how simples are combined) is given first of all by his interpretation of the arithmetic operations, including the extraction of roots, and then by his methods for constructing more algebraically complex problems on the basis of the solutions of simpler problems; in relation to that hierarchy, methods are then also given for constructing more algebraically complex curves on the basis of algebraically simpler curves.

But the assessment of Descartes' success must be nuanced, for in many respects he does not adhere to his own strict requirements for reduction; moreover, his reductionism inhibits his mathematical discoveries. In every one of his constructions, Descartes is forced to smuggle in entities (especially curves, and first and foremost the circle) that are not strictly licit at that stage of construction; his "starting points," the straight line segments, require the prior availability of curves as means of construction, just as later the construction of certain higher curves require as means of construction curves that are strictly speaking not yet available. His construction procedure itself turns out to be a combination of algebraic, geometrical, and mechanical means that cannot be contained by a linear "order of reasons."

The primary aim of the *Geometry* is to exhibit geometry as an ordered domain of construction problems that can be solved not by mere ingenuity, but by a method; algebra is important for the resources and the order it offers, but plays an auxiliary role. The simplest such problems are inherited from the Greek canon: for example, given two line segments, construct the line that is the mean proportional between them. The three famous problems of classical antiquity were the squaring of the circle, the duplication of the cube, and the trisection of the angle. The Greeks considered construction by means of ruler and compass (the intersections of lines and circles) to be the most rigorous and well-defined, and of course could not solve those three problems by such means. For the Greeks, acceptable "constructing curves" were primarily the circle and straight line; Descartes, by contrast, wished to generalize the very means of construction in an orderly way, and postulated a hierarchy of problems associated with constructing curves of increasing complexity. According to Descartes, a geometrical problem had to be translated into an algebraic equation in one unknown; then the roots of the equation had to be constructed by geometrically acceptable means: the "constructing curves" he allowed and catalogued included the conic sections and a few cubic curves, but he envisaged the use of higher algebraic curves.

The task of making precise what he meant by "increasing complexity" led Descartes to combine speculatively considerations about the properties of the curves themselves, the structure of the algebraic equations associated both with the problems and with the curves, and various novel mechanical devices for tracing curves that had interested him twenty years before. [9] Thus in the *Geometry* Descartes embarks on a program of classifying problems according to the complexity of the curves needed for their construction. He has first of all to explain which curves are acceptable as constructing curves, and then to find criteria according to which such curves could be set in order. Finally, he has to show how to choose the simplest constructing curves for a given sort of problem. Overall, he hoped to provide a universal method, using algebra to analyze problems, of finding the constructions for any problem that arose within the tradition of geometrical problem solving, as well as to identify and order all means beyond ruler and compass for these constructions. In the end, Descartes accomplished both more and less than he intended.

3. Descartes' *Geometry*: Diagrams as Procedures

On the very first pages of the *Geometry*, Descartes shows how the operations of addition, subtraction, multiplication, division, and the extraction of square roots can be interpreted geometrically, as operations on straight line segments that produce straight line segments. The "order of reasons" dictates that geometry begins with straight line segments, and proceeds to more complex entities by the following operations. Addition and subtraction are straightforward, but multiplication and division are not. (Figure 1) To multiply BD by BC, take AB as the unit; join the points A and C, and draw DE parallel to CA; then BE is the product of BD and BC. To divide BE by BD, take AB as the unit, join E and D, and draw AC parallel to DE; the segment BC is the quotient. Note that Descartes sets up two proportions, AB : BD :: BC : BE and BE : BD :: BC : AB, where AB = 1. This diagram thus represents a procedure for finding products and quotients, and as such is a schema for a simple tracing device.[10] In this capacity, it is symbolic: it represents something it does not resemble. However, the cogency of the procedure rests on Euclidean results that follow from the similarity of the two triangles ACB and DEB; in this capacity the diagram is iconic. And it must function as both to serve Descartes' purposes.

The same ambiguity holds for Figure 2, which explains how to find a line segment that represents the square root of a given line segment GH. To find the square root of GH, add to GH the line segment FG taken equal to the unit; bisect FH at K and describe the circle FIH around K taken as the center. Draw a perpendicular at G, intersecting the circle at I, and GI is the required root. [11] This diagram is symbolically the schema of a procedure and a tracing device, and iconically a circle and a right triangle divided into two similar right triangles. Descartes draws his inference because of the Euclidean result that the triangle FIH is similar to the triangles GFI and GIH, due to the way the circle constrains inscribed triangles (the angle FIH must be a right angle) and to the Pythagorean Theorem. The triangles and the circle are not present as objects of investigation; they are auxiliary constructions that help to advance the argument, not what the argument is about, that is, the relation between GH and the sought-for line segment GI.

All the same, they and the Euclidean results about them must be assumed to be available. The role of the circle in the diagram is both iconic and symbolic, and moreover its iconic role undermines Descartes' reductionist intention that we read the diagram as a configuration of straight line segments. Descartes fails to draw in the remaining sides of the triangles FIH, GFI and GIH; their absence testifies to Descartes' wish to downplay their iconic role. In sum, the construction of the square root requires similar triangles as well as a curve—a circle—and the construction of other roots will require other curves of higher degree. To get all the line segments he needs as representatives of rational and algebraic numbers, Descartes requires antecedently available curves to serve as means of construction; but he also want curves to be introduced as complex constructions derived from proportions holding among line segments and does not wish to admit them as starting points.

Descartes also never explains how the Greek theory of ratio and proportion based on similitude stands in rational relation to the algebra of arithmetic based on the equation. He simply posits the link between geometrical problems given in terms of ratios and proportions, and algebraic equations, explaining how a geometer can use the foregoing interpretation of the operations to derive an algebraic equation, whose solution will yield the solution to the problem. The construction of plane problems by means of algebra is exemplified in Figure 3, where the circle is represented by a dotted line. [12] A problem that can be constructed by means of ruler and compass can be expressed by a quadratic equation in one unknown: the square of an unknown quantity, set equal to the product of its root by some known quantity, increased or diminished by some other quantity also known, or $z^2 = az + b^2$. A solution of this equation can then be found by the following geometrical construction. The side LM of the right triangle NLM is equal to b (the square root of the known quantity b^2) and the other side LN is equal to half of a. Prolong MN to O, so that NO is equal to NL, and the whole line OM will be z, the required line and the root of the equation. A simple application of the Pythagorean Theorem shows that this construction is expressed by the formula: $z = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}$. [13] The construction, again, depends on both the prior availability of the circle as a constructing curve, and the validity of the Pythagorean Theorem. The circle is dotted because of its symbolic and auxiliary role.

4. Descartes' *Geometry*: Generalization to the Construction of a Locus

Having established his novel geometrical interpretation of the arithmetic operations, Descartes must then consider two different ways of generalizing his approach.

The first is the generalization to problems whose algebraic expression requires two variables, not just one. This he does by discussing Pappus' problem, a family of problems where what must be constructed is not just a single point (for example, O or I on the two preceding diagrams) and therefore a line segment determined by it, but a rather a locus, which must be constructed point-wise. The second, which may also be illustrated in terms of Pappus' problem, is the generalization to problems whose algebraic expression requires exponents higher than 2; when the degree of the equation is, for example, 3 or 4, there are general formulae for the roots, but they involve the extraction of cubic roots, and these cannot be constructed by ruler and compass.

Descartes was introduced to Pappus' problem in 1631 by a Dutch mathematician Jacob von Gool (Golius) who thought Descartes might want to try out his new method on it. In Book VII of his *Collectio*, Pappus of Alexandria (4^{th} c.) proposed the generalization of a problem that had been around since Euclid's time, and which implied a whole new class of curves. Since it was a problem that the Greek mathematicians could not solve in a methodical way nor properly generalize, it exhibited particularly well the power of Cartesian method. It asks for the determination of a locus whose points satisfy one of the conditions illustrated by Figure 4. Let the d_i denote the length of the line segment from point P to L_i which makes an angle of φ_i with L_i . Choose □α / β to be a given ratio and a a given line segment. The problem is to find the points P which satisfy the following conditions. If an even number ($2n$) of lines L_i are "given in position," the ratio of the product of the first n of the d_i to the product of the remaining n d_i should be equal to the given ratio α / β, where α and β are arbitrary line segments. If an odd number ($2n$ - 1) of lines L_i are "given in position," the ratio of the product of the first n of the d_i to the product of the remaining $(n - 1)$ d_i times a should be equal to the given ratio □α / β□ The case of three lines is exceptional, since it arises when two lines coincide in the four-line problem; the condition there is $(d_1 \cdot d_2) / (d_3)^2 =$ □α / β□ There are in fact points that satisfy each such condition, and they will form a locus on the plane. [14]

In the middle of Book I of the *Geometry*, Descartes announces, "I believe I have in this way completely accomplished what Pappus tells us the ancients sought to do," as if he had solved the problem in a thoroughgoing way for any number of lines. [15] While it is true that his combination of algebraic-arithmetical and geometrical devices produces an important advance in the solution of the problem, it is not true that his treatment of the problem is complete. In fact, there prove to be important and unforeseen complexities which arise with respect to geometrical curves, polynomial equations, and number itself, as well as the relations among them. Descartes' explanation of his method for solving the family of problems collected under Pappus' problem is given at the end of Book I, accompanied by a diagram of the four line version. (Figure 5) [16] He reduces the problem, which concerns proportions among lines "given in position"—as well as areas and volumes—and the loci they determine, to problems that are part of his geometrical program of the construction of equations, and so involve equations among line segments.

Treating the problem as an analysis in Pappus' sense, he depicts it as if it were already solved, and reduces the complexity of the diagram "by considering one of the lines given in position and one of those to be drawn (as, for example, AB and BC) as the principal lines" in terms of which all the other lines will be expressed. That is, all the lines must be labeled, and the segments whose lengths we know carefully distinguished from those we don't; and we must write down all the equations we can that express the relations between known and unknown segment lengths. Thus, he chooses *y* equal to BC (d_1) and *x* equal to AB, and then shows how all the other d_i can be expressed linearly in *x* and *y*. For 2*n* lines, the equation will be of degree at most *n*; for 2*n* - 1 lines, it will be of degree at most *n*, but the highest power of *x* will be at most *n* - 1. (For 2*n* and 2*n* - 1 parallel lines, where *y* is the sole variable involved, the result is an equation in *y* of degree at most *n*.) One highly significant feature of this diagram is that there is no locus depicted, only the point C which functions as its representative. By its minimally schematic depiction, the point C seems to promise that all curves that ought to belong to geometry can be understood as a nexus of line segments, which is of course the dogma of Descartes' reductionist method applied to mathematics.[17]

The point-wise construction of the locus is then undertaken as follows. One chooses a value for *y* and plugs it into the equation, thus producing an equation in one unknown, *x*. For the case of three lines, the equation in *x* is in general of degree 2; for 2*n* lines, it is of degree at most *n*; and for 2*n* - 1 lines, it is of degree at most *n* - 1. (For 2*n* - 1 parallel lines, the equation already has only one variable, *y*, and is of degree *n*.) The roots of this equation can then be constructed by means of intersecting curves which must be decided upon. This procedure generates the curve point by point and is thus potentially infinite, yet it also clearly does not generate all the real points on the line. It seems that Pappus' problem has been reduced to the geometrical construction of roots of equations in one unknown: the construction of line segments on the basis of rational relations among other line segments.

However, this reduction has an odd effect upon Descartes' own understanding of what he has accomplished. One of the first things that he says about Pappus' problem is that he has found a methodical way to classify cases of the problem; but his classification is based not on some feature of the locus generated but rather on what kind of constructing curve ought to be chosen in the point-wise construction of the locus, that is, in the construction of the line segment x given the relevant equation in x and y and a definite value of y. He sets out this classification of cases at the very end of Book I in more explicitly algebraic terms; it does not pertain to curves (describable by indeterminate equations in two unknowns) but to problems (describable by determinate equations in one unknown). Recall that Figure 5 contains no hint of the locus, but consists only of the nexus of line segments with their specified relations to an arbitrary point C of the locus. The official subject of these diagrams is line segments; constructing curves also intervene, but the choice of such curves is vexed for Descartes, and they are not what the diagram is about.

Book II, entitled "On the Nature of Curved Lines," attempts to classify curves, but it classifies them first of all in their role as means of construction for problems. Descartes begins by observing, "The ancients were familiar with the fact that the problems of geometry may be divided into three classes, namely, plane, solid, and linear problems. This is equivalent to saying that some problems require a conic section and still others require more complex curves. I am surprised, however, that they did not go further, and distinguish between different levels of these more complex curves, nor do I see why they called the latter mechanical, rather than geometrical." [18] Descartes criticizes the Greeks for failing to generalize their means of construction, and to subject them to rational constraint and methodical classification. In his view, the Greeks were rather haphazard in their approach, experimenting with the spiral and the quadratrix (curves which we now call transcendental, since they cannot be expressed by an algebraic equation) as constructing curves, while at the same time failing to generalize on ruler and compass.

5. Descartes' *Geometry*: Generalization to Higher Algebraic Curves

We might expect Descartes to make his generalization simply on the basis of the algebraic equation. But at the beginning of Book II, rather than invoking equations Descartes appeals to tracing machines. For he understands his project not only as the exploration of the new algebra in the service of geometrical problem-solving, but also, and primarily, as the geometrical construction of the roots of algebraic equations, the discovery of line segments on the basis of a given configuration of line segments. Moreover, as Descartes knew very well, while the point-wise construction of loci articulated in Book I might look like a good way to generate more and more complex curves starting from rational relations among line segments, it is not a satisfactory way to generate curves that will be used as constructing curves. The indefinitely iterated, point-wise construction of loci does not guarantee the existence of all the points of intersection required when curves are used as means of construction; stronger continuity conditions are required. [19]

It is instructive to compare the classification of cases that occurs at the end of Book I, which seems rather more algebraic, and the classification of curves by genre at the beginning of Book II, which invokes tracing machines. The classification at the end of Book I stems from a prior classification of problems: when three or four lines are given in position, the required point may be found by using circle and straight line; when there are five, six, seven, or eight lines given in position, the required point may be found by a curve "of next higher level," which includes the conic sections; and when there are nine, ten, eleven, or twelve lines, the required point may be found by a curve of "next highest level," though it is unclear what that level includes. Descartes' definition of genre occurs right in the midst of his discussion of two tracing machines in the first pages of Book II. He explains why they are acceptable generalizations of ruler and compass, and therefore likely sources of constructing curves for the higher levels of problems. Descartes' first tracing machine (Figure 6) is a system of linked rulers that allows the user to find one, two, three, or more mean proportionals between two given line segments. This is, clearly, an ordered series of problems. As it is opened, the machine traces out certain curves AD, AF, AH, and so on, which then function as constructing curves, since their intersections with the circles determine the sought-for line segments, the mean proportionals. Descartes recognizes that these constructing curves form a series of higher algebraic curves of increasing complexity, though he does not give an equation for any of them; nor does he explain the relation between this series of problems and the series of problems united under Pappus' problem. Descartes cannot rely on his tracing machine alone to generate a complete series of problems and constructing curves; the series they generate is too special. His hinged rulers do not, for example, produce the curve he is most interested in exhibiting as the fruit of his method, the "Cartesian parabola," which is not a parabola at all but a cubic curve.

Descartes does trace the Cartesian parabola by using another tracing machine (Figure 7), which generates new, more complex curves from the motion of simpler curves and straight lines. A ruler GL is linked at L to the device NKL, which can be moved along the vertical axis while the direction of KN is kept constant; as L slides up or down the vertical axis, GL turns around G, and the line KN is moved downwards remaining parallel to itself. The intersections of the line KNC and GL trace out a curve GCE, which is represented by a dotted line. When the line KN is a straight line, GCE is a hyperbola, whose equation Descartes derives. When KN is a circle GCE is a conchoid; when KN is a hyperbola, GCE is the Cartesian parabola, for which he also derives an equation. Once again there is a series of constructing curves, and once again it is rather special. Descartes cannot guarantee that all the curves needed to trace Pappian loci, or the loci themselves, can be traced by such machines. Indeed, later on in Book II, in the second case of the five line locus problem, Descartes describes a curve which he did not know how to trace by continuous motion. [20]

In sum, the point-wise construction of the Pappian loci do not give enough points to underwrite those curves as constructing curves; the tracing machines guarantee continuity, but produce series that are too special and incompletely understood algebraically; and in any case, though Descartes believed that all algebraic curves could be described as Pappian loci, this is in fact not true. Moreover, not all tracing procedures are acceptable to Descartes. He distinguishes between "geometrical" and "mechanical" curves, a distinction that corresponds to the modern distinction between algebraic and transcendental curves. The former are all and only those that correspond to algebraic equations; the latter are not acceptable in geometry. Yet transcendental curves can be traced by well specified tracing methods. Since in his exposition he affirms that tracing procedures supplement algebraic approaches, Descartes must give a demarcation criterion. At the beginning of Book II, Descartes rejects the spiral and quadratrix (two transcendental curves) by specifying that they belong not to geometry but to mechanics, "because we imagine them described by two separate motions, which have no relation between them which can be precisely measured." [21] Thus, in the tracing motions Descartes finds acceptable, the tracing point is the intersection of two moving straight or curved (algebraic) lines, and the motions of the lines are continuous and strictly coordinated by an initial motion. However, exactly what Descartes intended by continuous motion and strict coordination is never really spelled out in the *Geometry*. Ultimately, Descartes' segregation of algebraic from transcendental curves (and his belief that the coordination of moving lines in the tracing of the latter is not strict) rests on his belief that straight and curved lines do not stand in rational relation to each other, a belief that underwrites his reductive method for geometry, where everything in the "order of reasons" is built up from the concatenation of straight line segments. [22]

So when Descartes announces, "Having now made a general classification of curves, it is easy for me to demonstrate the solution that I have already given of the problem of Pappus," he is overstating his case. Referring to Figure 8, which is reprinted no less than four times in the pages that follow, he explains under what conditions the locus will be a circle (as it is depicted in Figure 8), a parabola, a hyperbola, and an ellipse, and what features its associated equation will have. Note that only the four lines "given in position" are printed as continuous lines; all the others, including the locus, are printed as dotted lines. As a solution to this problem, the locus is constructed point-wise; it is a sort of compendium of solutions to an indefinitely infinite series of problems about relations among straight line segments, and so is the associated algebraic equation. They both sum up and exhibit that set of problems very nicely, and in that capacity they are both symbolic rather than iconic. As a symbolic representation of that set of problems, however, the circle is not continuous.

Thus when Descartes presents his most novel curve, the Cartesian parabola—the curve that he discovered and investigated—he does not only present it as a locus, that is, as a solution to Pappus' problem. He also presents it as the result of a tracing machine. Figure 9 exhibits the Cartesian parabola as a locus, a solution to Pappus' problem when there are five lines and four of them are parallel with the fifth perpendicular to the first four. In addition, the second tracing machine (Figure 7) is superimposed on the representation of Pappus' problem, and it is labeled so as to underscore the superposition. The ruler GL in Figure 7 corresponds to the line GL in Figure 9; the line CNK in Figure 7 (which, Descartes tells us on the following page, can be replaced by various conic sections in order to trace out other curves) corresponds to the half-parabola KN; the line traced out in Figure 7 GCE corresponds to the Cartesian parabola CEG. Note that the parabola CKN is printed as a dotted line, for its role is a constructing curve, but the Cartesian parabola CEG is a continuous line. Descartes presents his new discovery as a Pappian locus, the result of a tracing machine, and as an algebraic equation, $y^3 - 2ay^2 - a^2y + 2a^3 = axy$. Three distinct modes of representation, related in the natural language exposition of the text, are needed for the inauguration of this novel curve, to indicate what it is as well as to begin the process of analyzing it.

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Notes

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[1] See, for example, J. J. Gray, "The nineteenth-century revolution in mathematical ontology," in D. Gillies, ed., *Revolutions in Mathematics*, Oxford: Clarendon Press, 1992, pp. 226-248.

[2] D. Hilbert, *Geometry and the Imagination*, op. cit., p.

[3]See, for example, the discussion in W. M. Boothby, *An Introduction to Differentiable Manifords and Riemannian Geometry*, New York: Academic Press, 1975, pp. 1-5.

[4] For further discussion of this point, see my "Constructive Ambiguity in Mathematical Reasoning," *Mathematical Reasoning and Heuristics*, D. Gillies and C. Cellucci, eds., King's College Publications, 2005, pp. 1-23; and "Leibniz on Mathematics and Representation: Knowledge through the Integration of Irreducible Diversity," in *The Young Leibniz*, M. Kulstad, ed., *Studia Leibnitiana* Sonderheft, forthcoming.

[5] These issues are discussed at length in Ch. 1 and 2 of my *Cartesian Method and the Problem of Reduction*, Oxford: Clarendon Press, 1991. The phrase "order of reasons" is taken form M. Gueroult, *Descartes selon l'ordre des raisons*, Paris: Aubier, 1968, 2 vol.

[6] See the opening pages of M. Gueroult's *Descartes selon l'ordre des raisons*, op. cit.

[7] *Descartes selon l'ordre des raisons* , op. cit., v. 1, Ch. 1.

[8] G 297, AT 369.

[9] See the interesting discussion in Chikara Saskaki's *Descartes's Mathematical Thought*, Dordracht: Kluwer, 2003, Ch. 3.2.

[10] I have never read an explanation of how this diagram originated. Perhaps Descartes, frustrated in his attempt to develop geometry algebraically at the end of the *Regulae*, realized that his earlier experiments with different kinds of tracing devices to construct points was the key to a more effective algebraization.

[11] Ibid.

[12] G 302-303, AT 374-375.

[13] G 297-303, AT 369-75.

[14] See H. J. M. Bos' discussion of Pappus' problem in his "On the Representation of Curves in Descartes' *Géométrie*" in *Archive for History of Exact Sciences* 24 (1981), pp. 298-302, from which this diagram (Figure 5.7) is derived.

[15] G 309, AT 382.

[16] G 310-12, AT 382-4.

[17] Bos, op. cit.

[18] G 315, AT 388.

[19] H. J. M. Bos, *Redefining geometrical exactness: Descartes' transformation of the early modern concept of construction*, Frankfurt: Springer-Verlag, 2001, Ch. 24.

[20] Ibid, Ch. 16, 23, 24.

[21] G 317, AT 390.

[22] *Redefining geometrical exactness: Descartes' transformation of the early modern concept of construction*, op. cit., Ch. 28.